Efficient Rare-event Simulation for Multiple Jump Events of Heavy-tailed Lévy Processes with Infinite Activities

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Northwestern University

Introduction

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Rare-event simulation for Lévy processes

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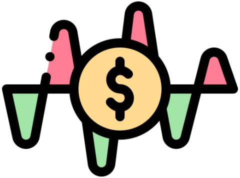
Risk analysis



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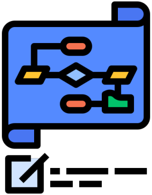

Queuing networks

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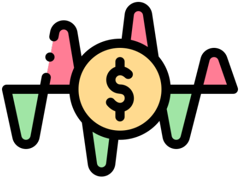

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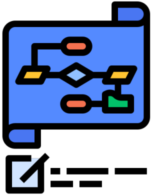

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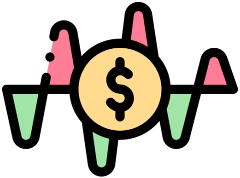

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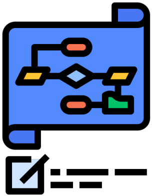

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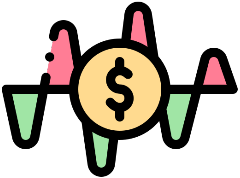

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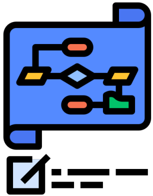

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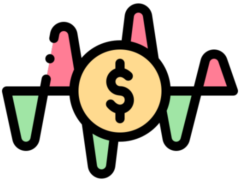

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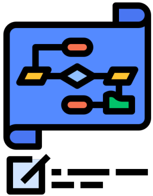

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- **Why is this difficult?**

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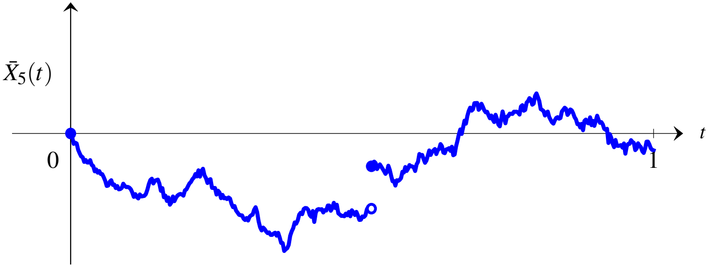

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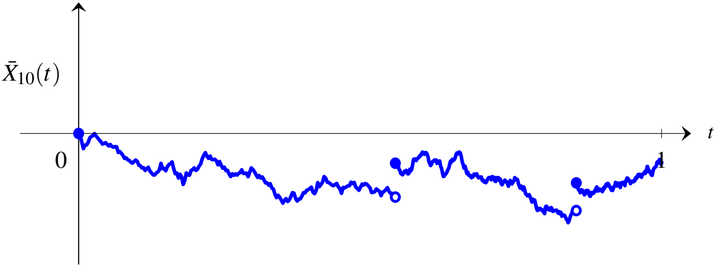

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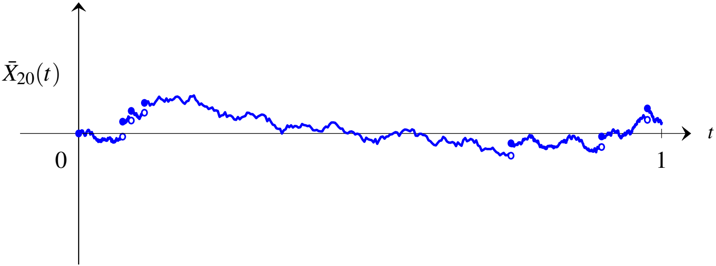

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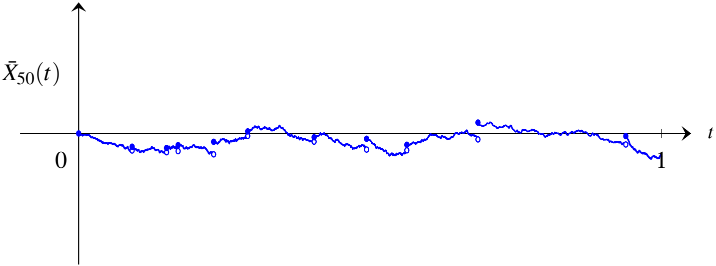

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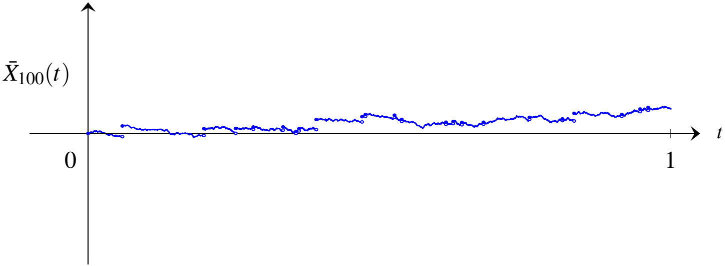

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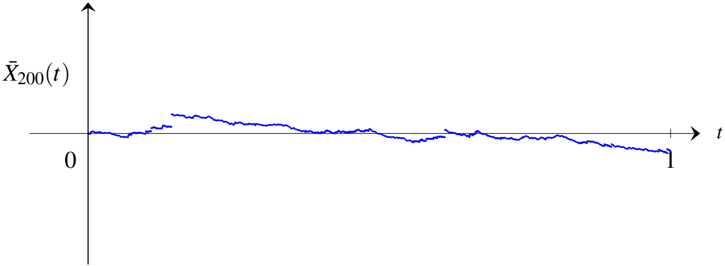

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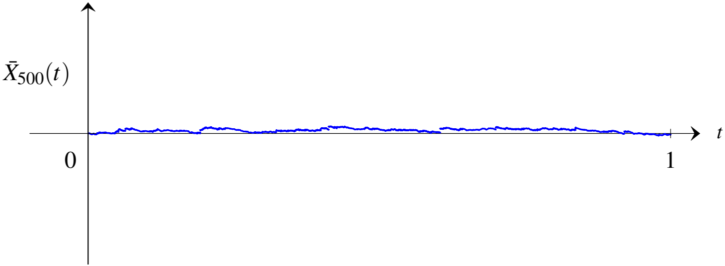

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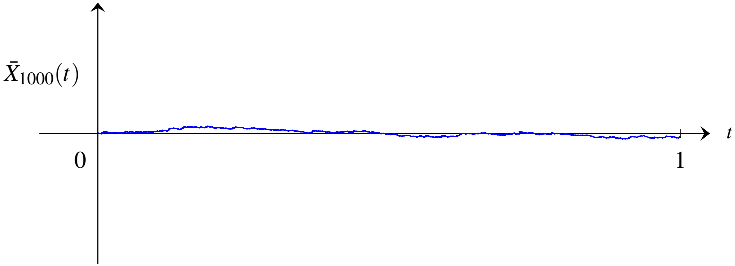

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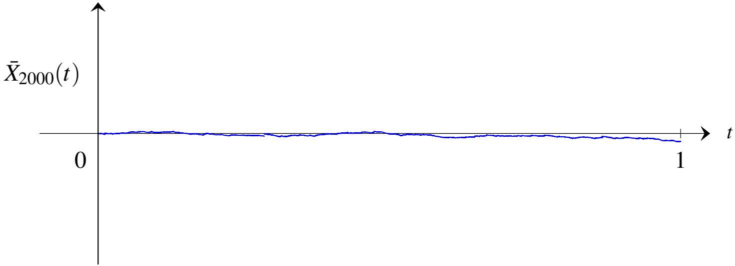

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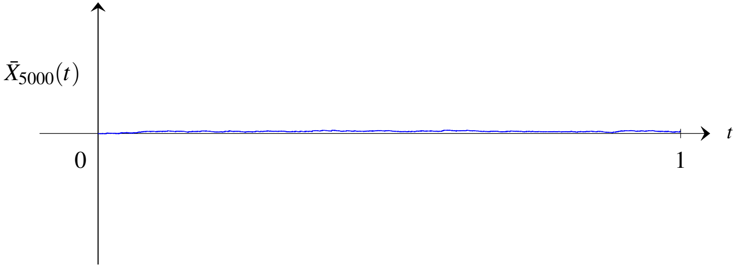

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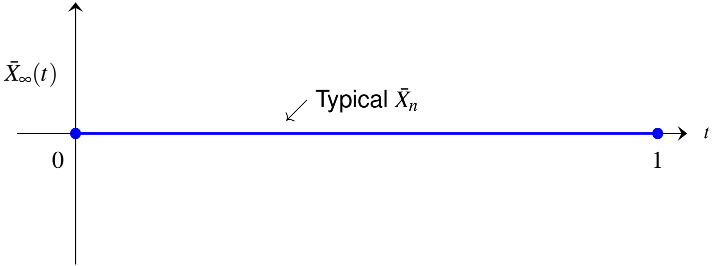

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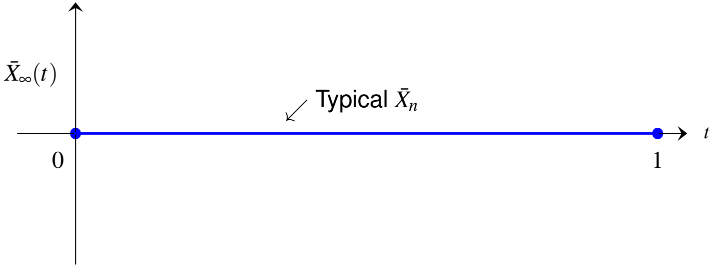

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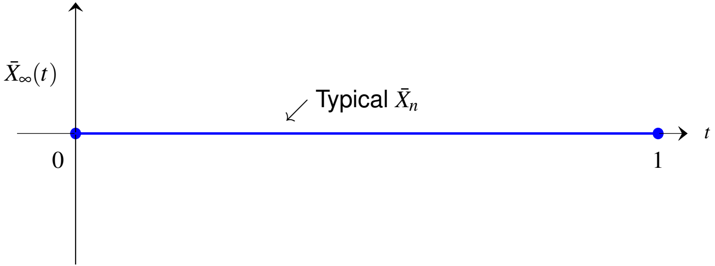

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↖Uniform bound for any n

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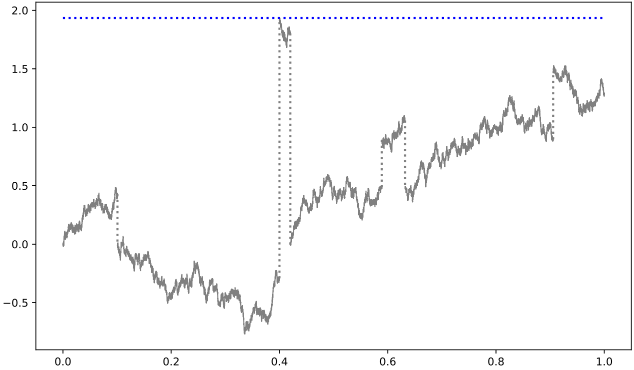

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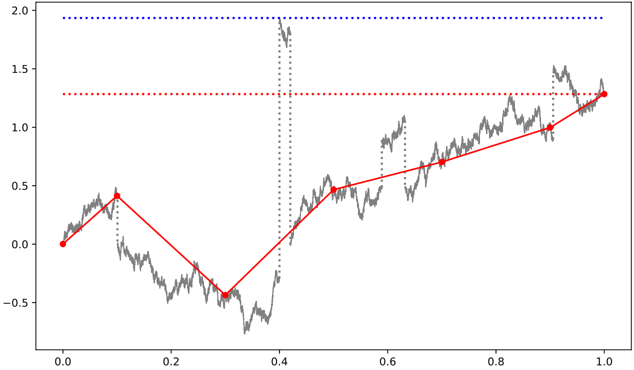

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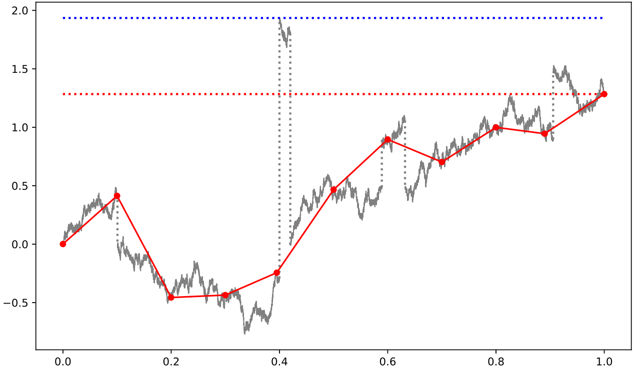

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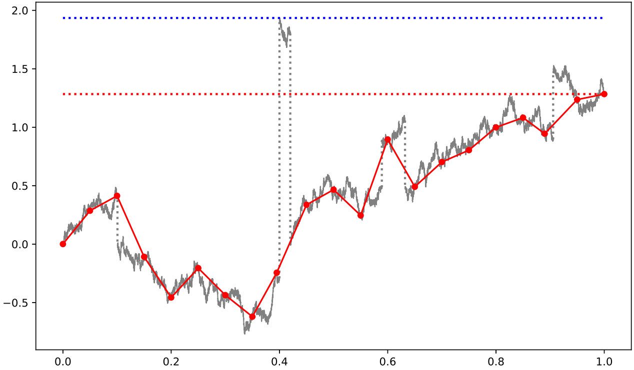

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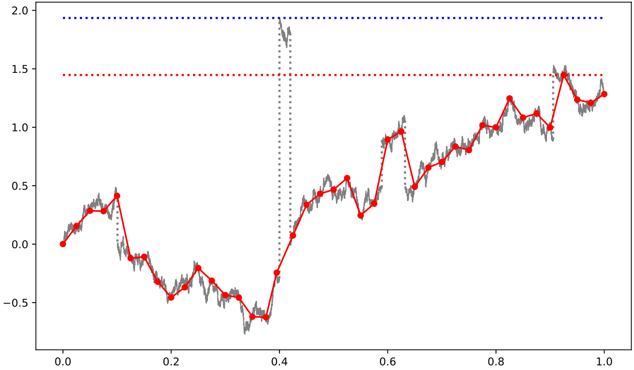

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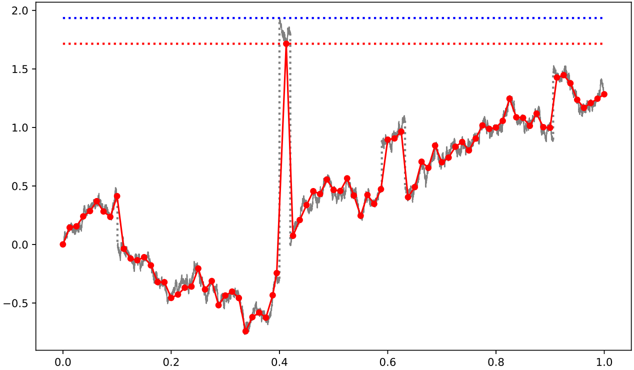

Algorithm

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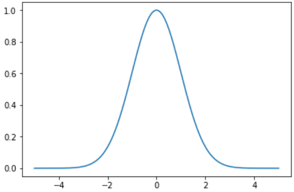
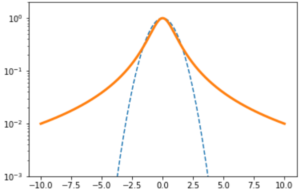


Gaussian



Heavy-tailed

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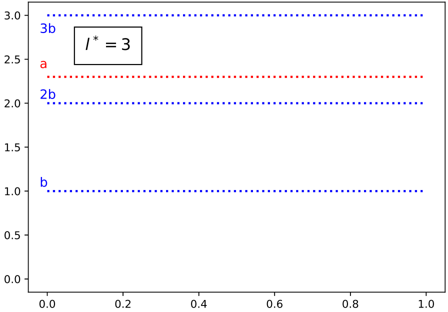

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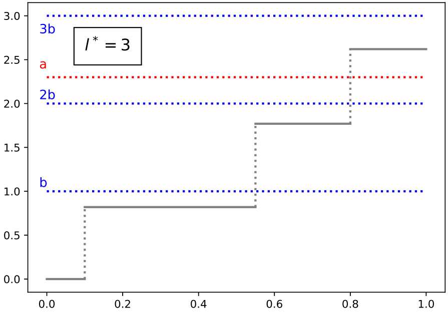

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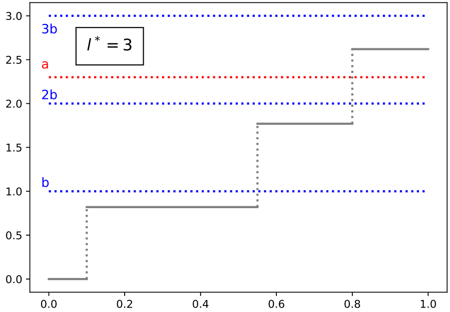

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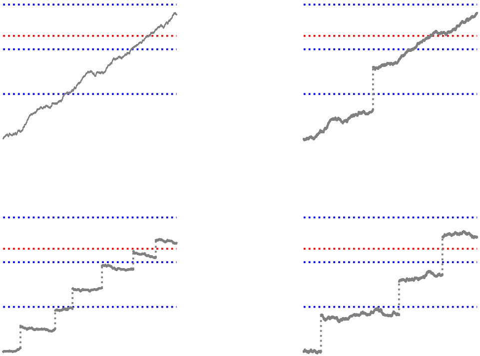

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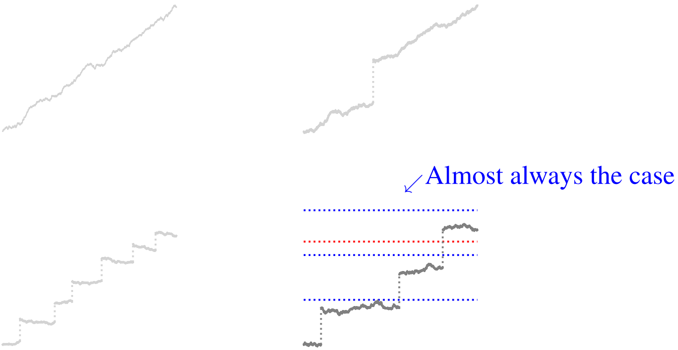

Large Deviation Principle for Heavy-tailed Lévy Processes

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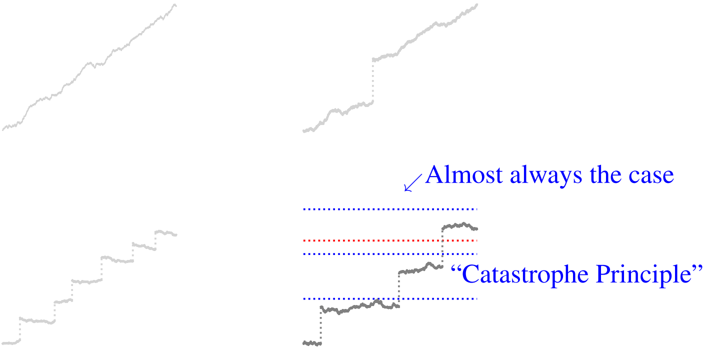

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We proposed algorithm to sample from $\mathbb{P}(\cdot|\bar{X}_n \in B)$

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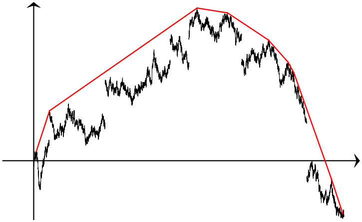
- **Question:** What approximation $(Y_m)_{m\geq 1}$ should we use?

Efficient Approximation via Concave Majorization

Concave majorant of X



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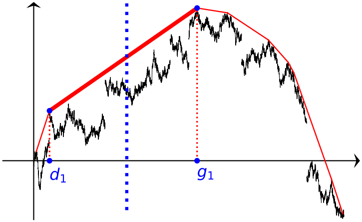

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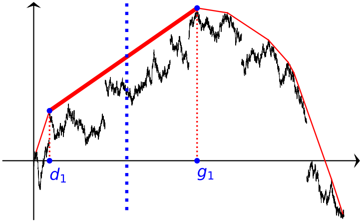

$l_1 = g_1 - d_1, \ \ s_1 = X(g_1) - X(d_1)$

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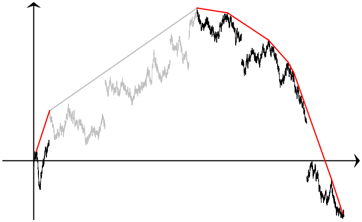

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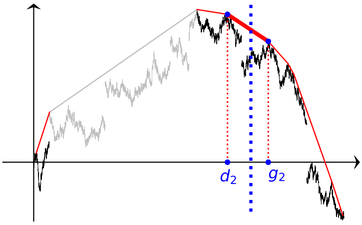

$l_2 = g_2 - d_2, \;\; s_2 = X(g_2) - X(d_2)$

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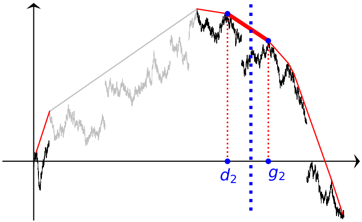

$\left(l_n, s_n\right)_{n\geq 1}$

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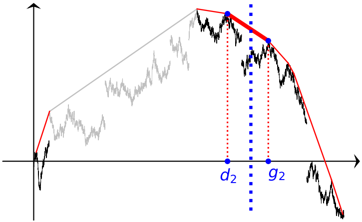

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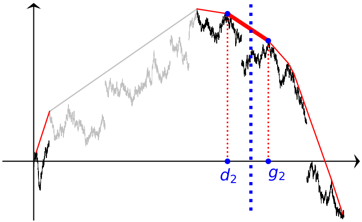

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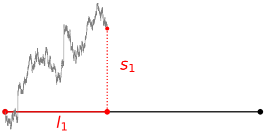

$l_1 \sim \text{Unif}(0,1)$

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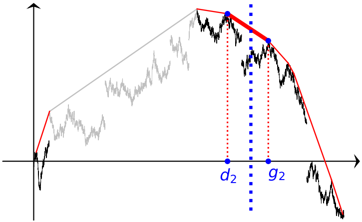

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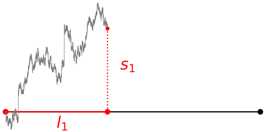

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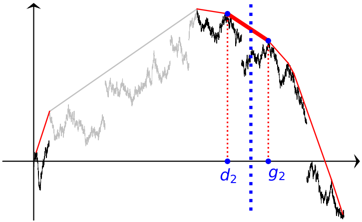

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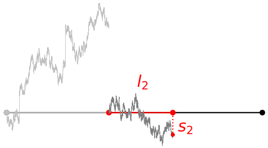

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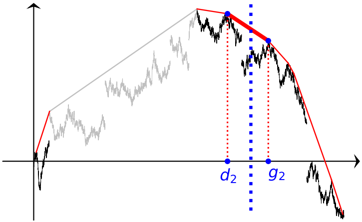

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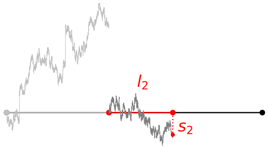

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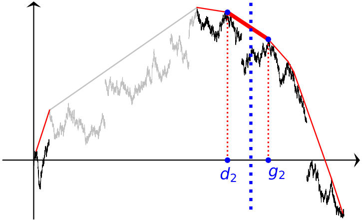

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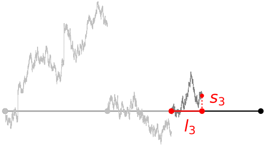

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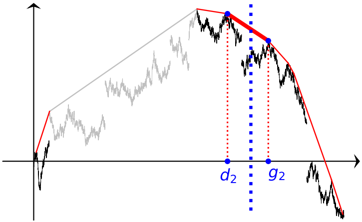

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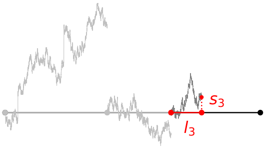

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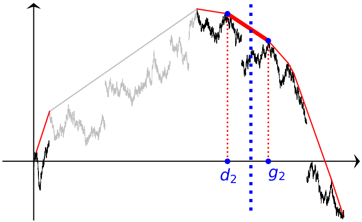

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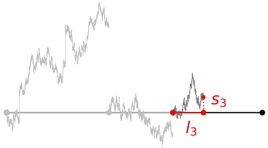

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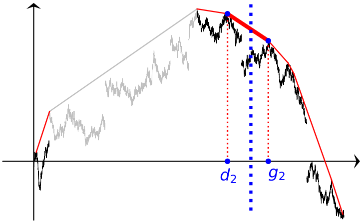

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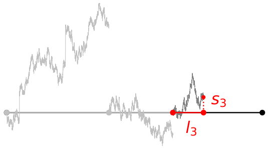

$$(l_n, s_n)_{n \geq 1} \overset{d}{=} (l_n, s_n)_{n \geq 1}$$

Pitman and Bravo (2012)

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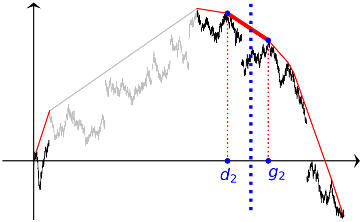

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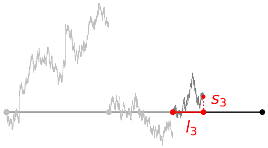
Stick breaking **approximation**



$\left(l_n, s_n\right)_{n=1}^{M}$

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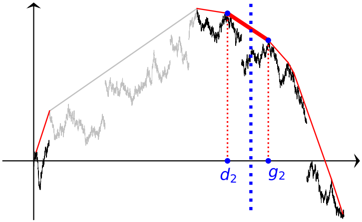

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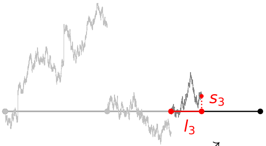

Geometric convergence rate↗

$\left(l_n, s_n\right)_{n=1}^{M}$

Theoretical Analysis of the Algorithm

Strong Efficiency

Algorithm 1 Efficient Estimation of $\mathbb{P}(A_n)$

Require: $w \in (0,1), \gamma > 0, \rho \in (0,1)$

if $\text{Unif}(0,1) < w$ **then** ▷ Sample J_n from $\mathbb{Q}$
  Sample $J_n = \sum_{i=1}^{k} z_i \mathbb{1}_{[u_i,n]}$ from $\mathbb{P}$
else
  Sample $J_n = \sum_{i=1}^{k} z_i \mathbb{1}_{[u_i,n]}$ from $\mathbb{P}(\cdot|B_n^\gamma)$ using Algorithm 2
end if
Let $u_0 = 0, u_{k+1} = n$.
Sample $\tau \sim \text{Geom}(\rho)$ ▷ Decide Truncation Index τ
for $i = 0,1,\cdots,k$ **do** ▷ Generate Stick Lengths, and Decide Increments
  Sample $U_1^{(i)} \sim \text{Unif}(0,1)$. Let $l_1^{(i)} = U_1^{(i)}(u_{i+1} - u_i)$
  Sample $\xi_{i,1} \sim F_{\widetilde{X}}(\cdot, l_1^{(i)})$
  for $j = 2,3,\cdots,\lceil \log_2(n^2) \rceil + \tau$ **do**
    Sample $U_j^{(i)} \sim \text{Unif}(0,1)$. Let $l_j^{(i)} = U_j^{(i)}(u_{i+1} - u_i - l_1^{(i)} - l_2^{(i)} - \cdots - l_{j-1}^{(i)})$
    Sample $\xi_{i,j} \sim F_{\widetilde{X}}(\cdot, l_j^{(i)})$
  end for
  Let $l^{(i)}_{\lceil \log(n^2) \rceil + \tau + 1} = u_{i+1} - u_i - l_1^{(i)} - l_2^{(i)} - \cdots - l^{(i)}_{\lceil \log(n^2) \rceil + \tau}$
  Sample $\xi_{i,\lceil \log_2(n^2) \rceil + \tau + 1} \sim F_{\widetilde{X}}(\cdot, l^{(i)}_{\lceil \log_2(n^2) \rceil + \tau + 1})$
end for
for $m = 0,1,\cdots,\tau$ **do** ▷ Evaluate $Y_{n,m}$
  for $i = 0,1,2,\cdots,k$ **do**
    Let $\widetilde{M}_m^{(i)} = \sum_{l=0}^{i-1} \sum_{j=1}^{\lceil \log_2(n^2) \rceil + \tau + 1} \xi_{l,j}^m + \sum_{j=1}^{\lceil \log_2(n^2) \rceil + \tau} (\xi_{i,j}^m)^+$
  end for
  Let $Y_{n,m} = \mathbb{1}\{ \max_{i=0,1,\cdots,k} \widetilde{M}_m^{(i)} + J_n(u_i) \geq na \}$
end for
Let $Z_n = Y_{n,0} + \sum_{m=1}^{\tau} (Y_{n,m} - Y_{n,m-1}) / \rho^{m-1}$ ▷ Return the Estimator L_n
if $\max_{i=1,\cdots,k} z_i > b$ **then**
  Return $L_n = 0$.
else
  Let $\lambda_n = n\nu[n\gamma, \infty)$, $p_n = 1 - \sum_{l=0}^{l^*-1} e^{-\lambda_n} \frac{\lambda_n^l}{l!}$, $I_n = \mathbb{1}\{J_n \in B_n^\gamma\}$
  Return $L_n = Z_n / (w + \frac{1-w}{p_n} I_n)$
end if

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Theorem (W. and Rhee, 2020)

If **assumption (A1)** holds, then our importance sampling algorithm is unbiased and strongly efficient.

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Assumption: For any $z_0 > 0$, there exist $C > 0, \beta > 0, \theta \in (0,1]$ such that for any $t > 0, z \geq z_0, x \in \mathbb{R}, \delta \in [0,1]$, we have

$$\mathbb{P}(X^{<z}(t) \in [x, x+\delta]) \leq \frac{C}{t^\beta \wedge 1}\delta^\theta; \tag{A1}$$

where $X^{<z}$ is the Lévy process with the generating triplet $(c_X, \sigma^2, \nu|_{(-\infty,z)})$.

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Theorem (W. and Rhee, 2020)

If **assumption (A1)** holds, then our importance sampling algorithm is unbiased and strongly efficient.

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X with infinite activities↗ ↖Only requirement: not too slow

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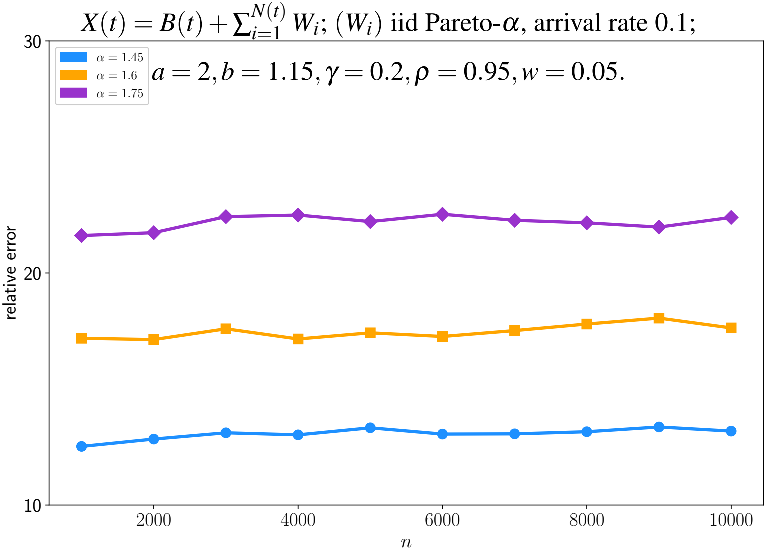
The algorithm is applicable to a broad class of heavy-tailed Lévy processes.

Numerical Experiments

Experiment Results: Reinsurance Case

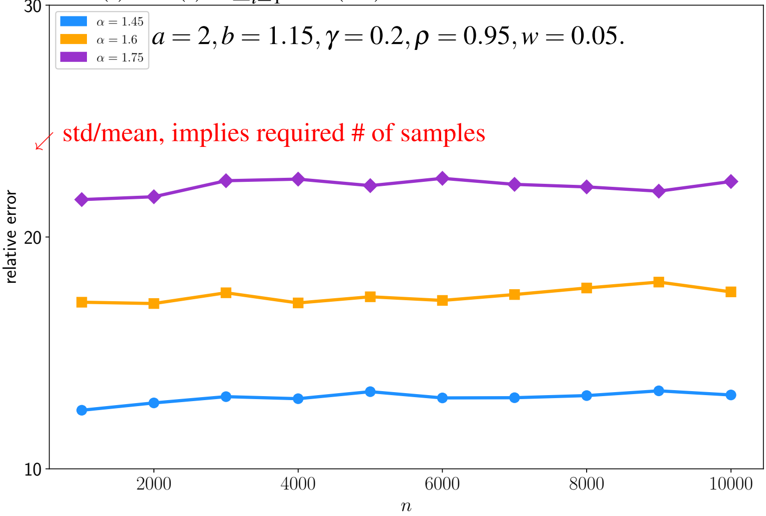


Experiment Results: Reinsurance Case

$X(t) = B(t) + \sum_{i=1}^{N(t)} W_i$; (W_i) iid Pareto-α, arrival rate 0.1;

$a = 2, b = 1.15, \gamma = 0.2, \rho = 0.95, w = 0.05.$

std/mean, implies required # of samples



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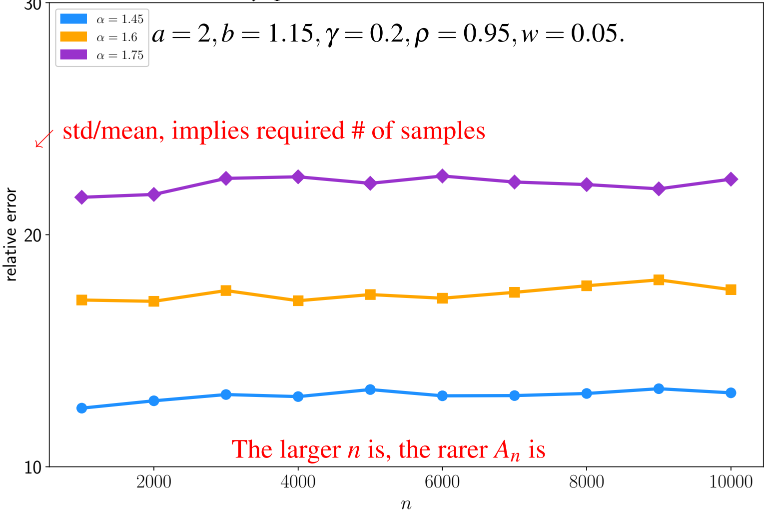

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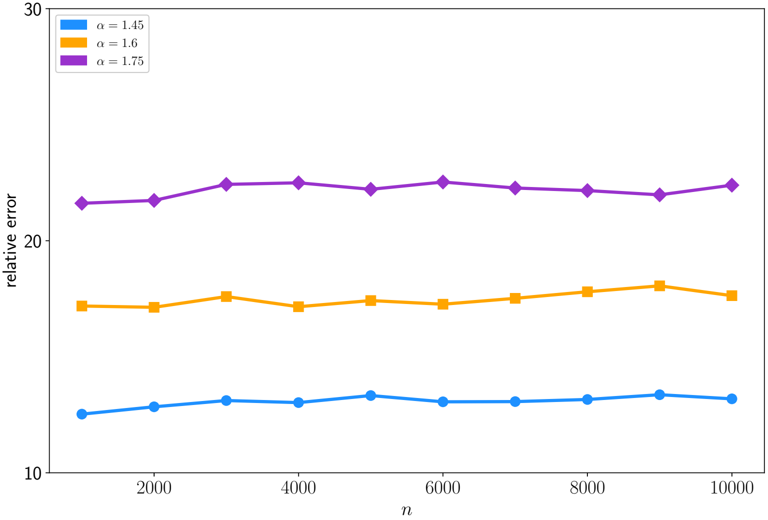

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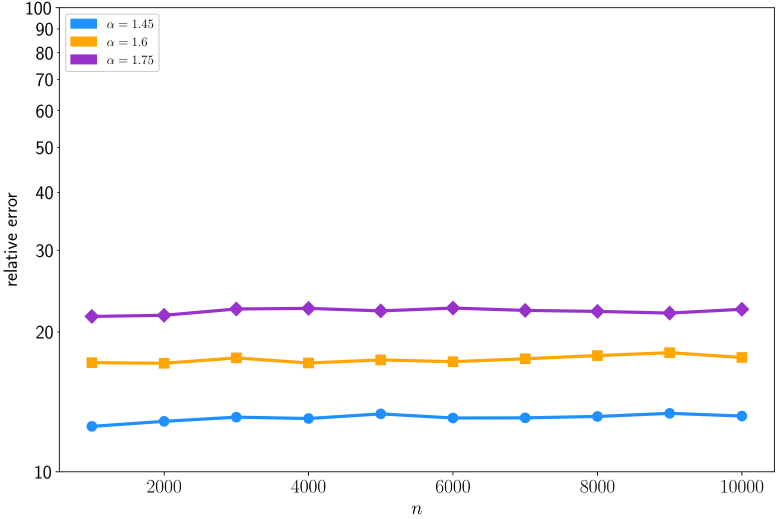

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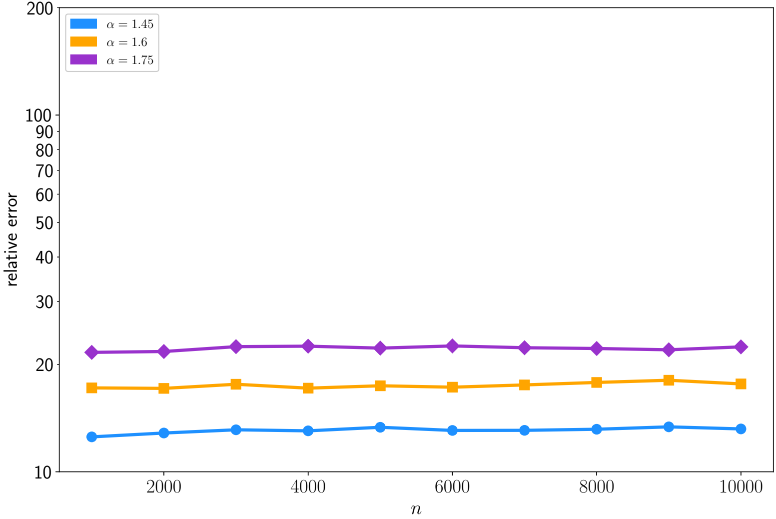

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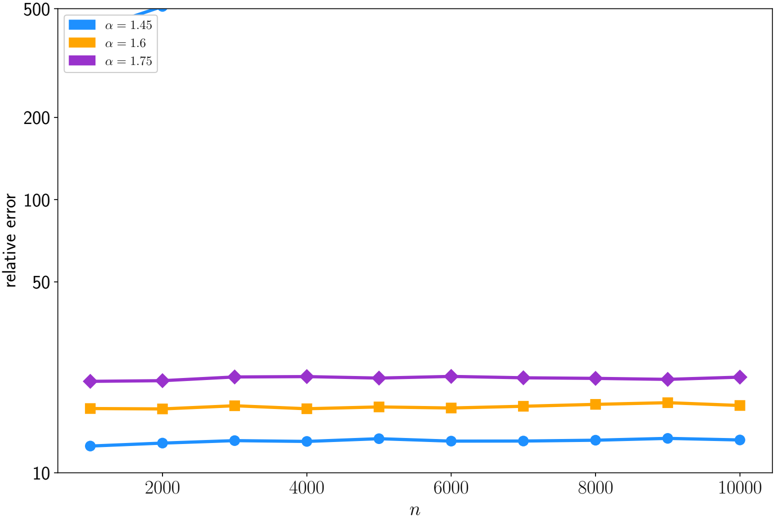

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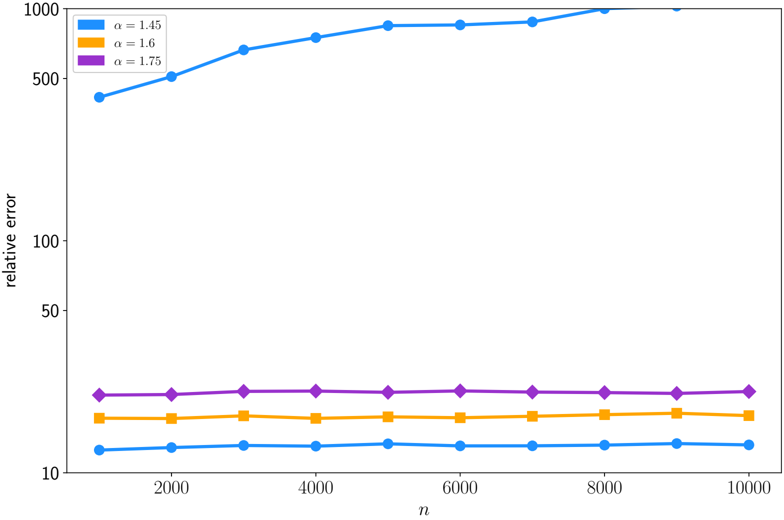

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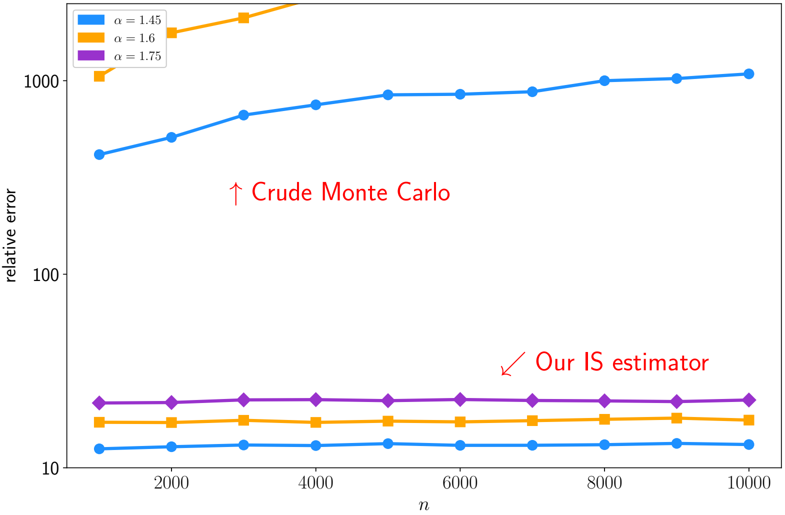

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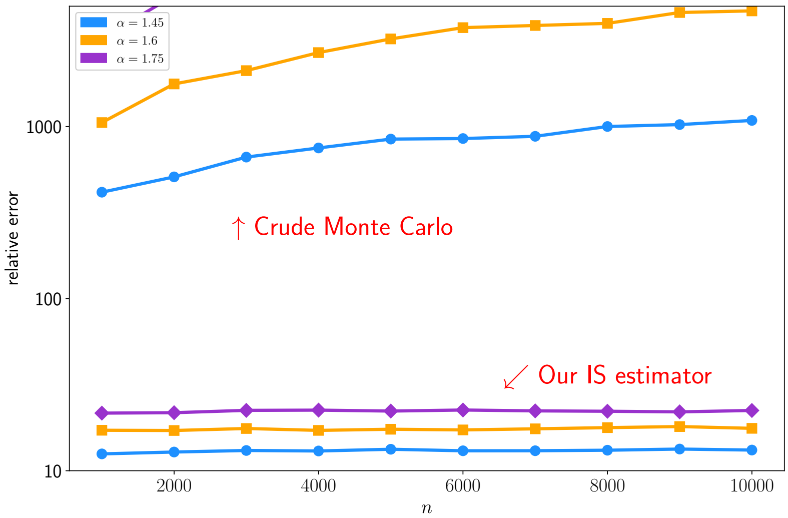

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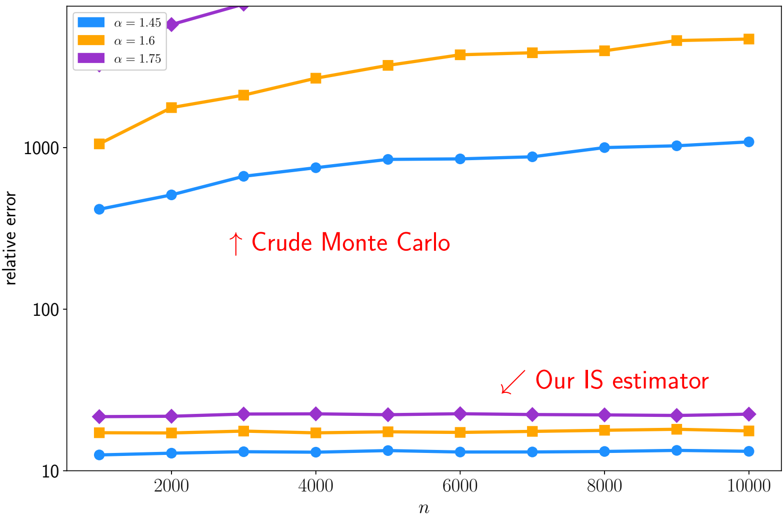

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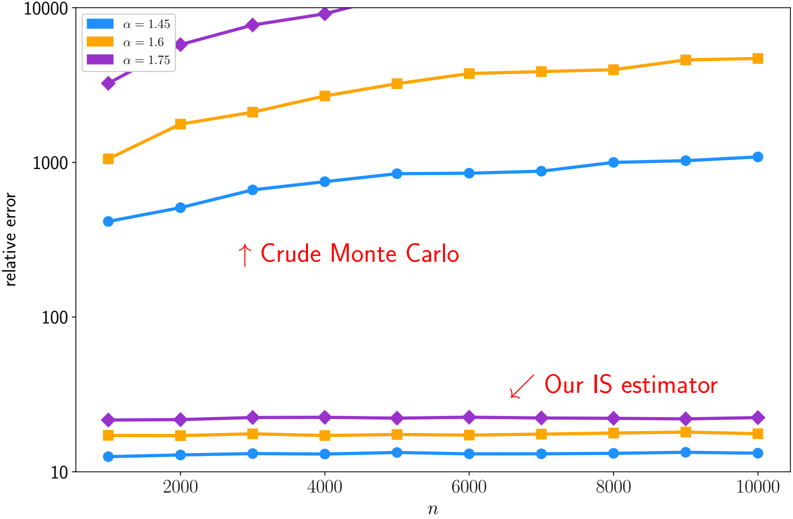

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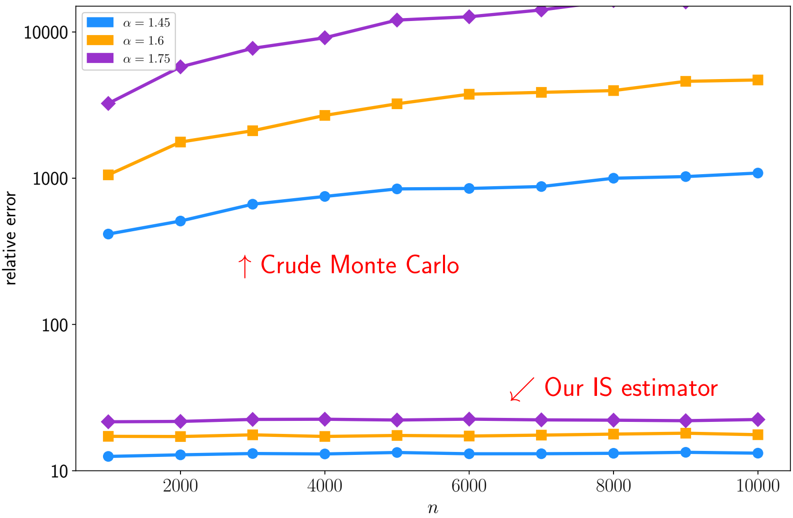

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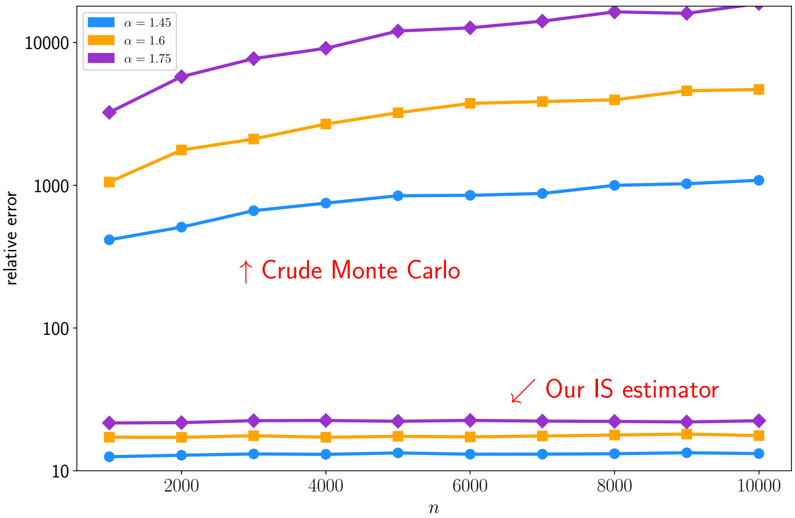

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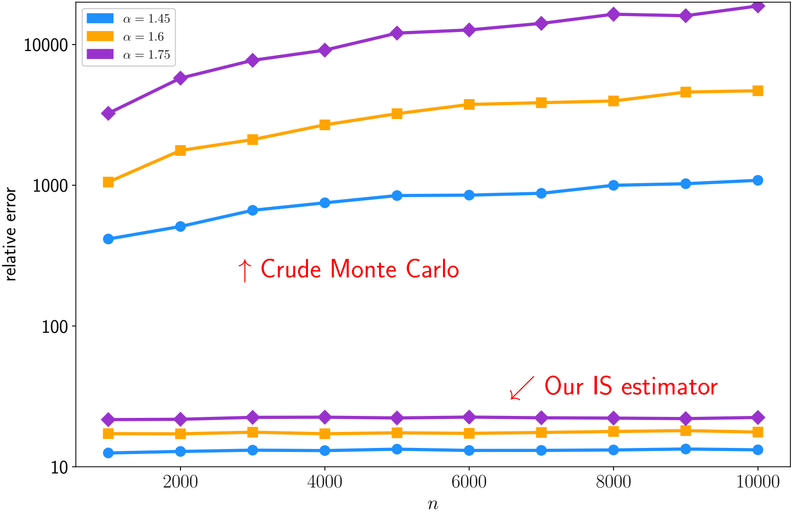

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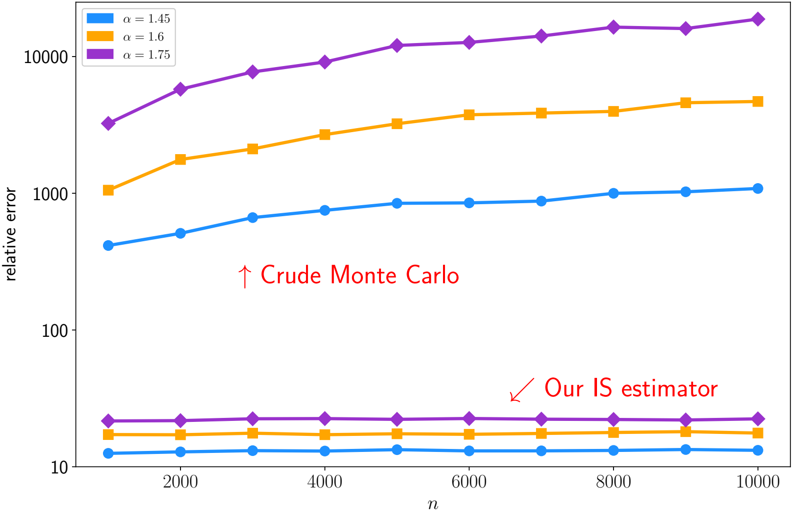

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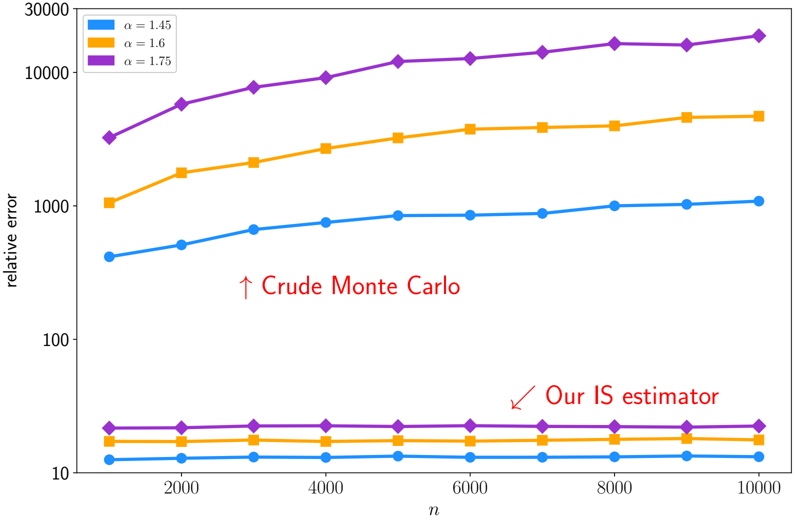

Conclusion

Proposed importance sampling algorithms

- for heavy-tailed Lévy processes with **infinite activities**
- with guarantee of **strong efficiency**
- **Significant improvements** illustrated in numerical experiments
- extension to cases where $X(t)$ is not simulatable is also available